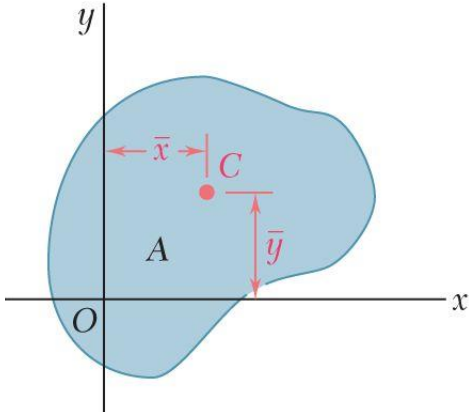
Chapter 10: Moments of Inertia

Recap from last chapter: First moment of an area (centroid of an area)

- The first moment of the area *A* with respect to the x-axis is given by $Q_x = \int_A y\,dA$

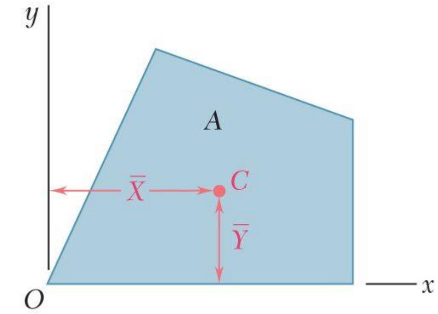
- The first moment of the area *A* with respect to the y-axis is given by $Q_y = \int_A x\,dA$

- The centroid of the area *A* is defined as the point *C* of coordinates $\bar{x}$ and $\bar{y}$, which satisfies the relation

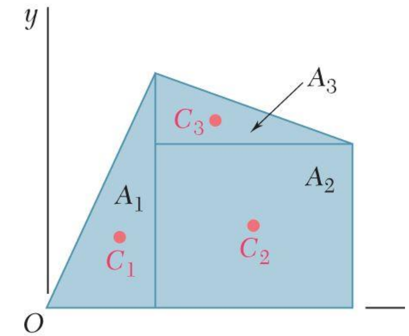


$$\int_A x\,dA = A\,\bar{x}$$

$$\int_A y\,dA = A\,\bar{y}$$



- In the case of a composite area, we divide the area *A* into parts A_1, A_2, A_3 =



$$A_{total}\,\bar{X} = \sum_i A_i\,\bar{x}_i \qquad A_{total}\,\bar{Y} = \sum_i A_i\,\bar{y}_i$$

Brief tangent about terminology: the term **moment** as we will use in this chapter refers to different “measures” of an area or volume.

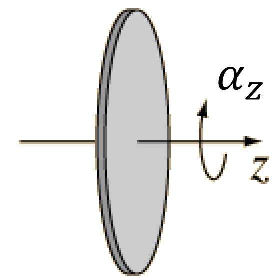
- The *first* moment (a single power of position) gave us the centroid.
- The *second* moment will allow us to describe the “width.”
- An analogy that may help: in *probability* the first moment gives you the mean (the center of the distribution), and the second is the standard deviation (the width of the distribution).

Mass Moment of Inertia

Mass moment of inertia is the mass property of a rigid body that determines the torque T needed for a desired angular acceleration (α) about an axis of rotation (a larger mass moment of inertia around a given axis requires more torque to increase the rotation, or to stop the rotation, of a body about that axis).
Mass moment of inertia depends on the shape and density of the body and is different around different axes of rotation.

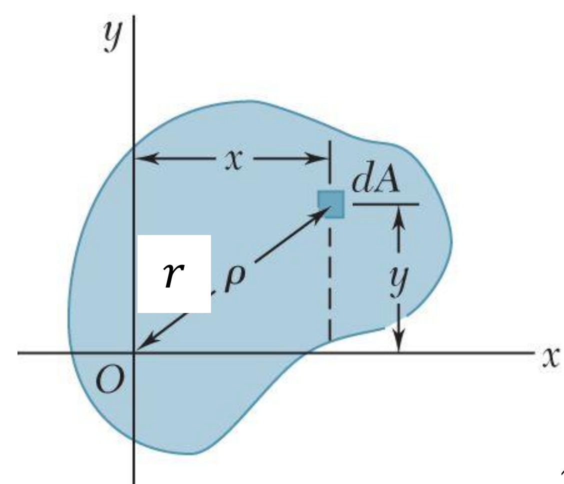
Torque-acceleration relation: $T = I\,\alpha$

where the mass moment of inertia is defined as $I_{zz} = \int \rho\, r^2 dV$



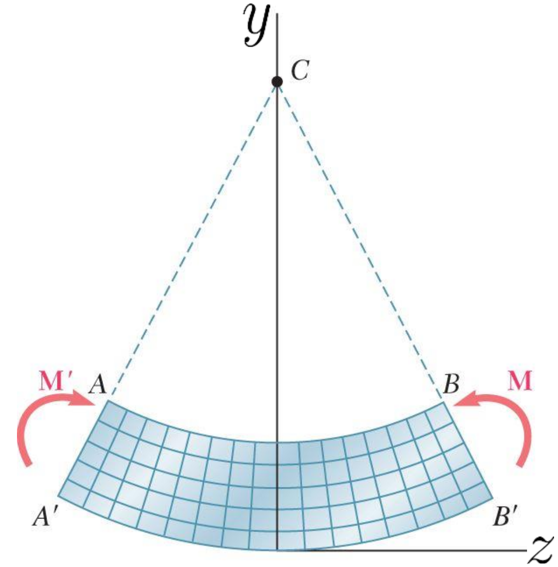
Mass moment of inertia for a disk:

$$
\begin{aligned}
I_{zz} &= \int \rho\, r^2\, dv = \int_0^t \int_0^{2\pi} \int_0^R \rho\, r^2\,(r\,dr\,d\theta\,dz) \\
&= \rho \int_0^t \int_0^{2\pi} \frac{r^4}{4} d\theta\, dz \\
&= \rho \int_0^t \frac{r^4}{2}\pi\, dz = \rho \frac{r^4}{2}\pi\, t = \frac{r^2}{2}\rho\,\pi\, r^2\, t = \frac{r^2}{2}\rho V = \frac{r^2}{2} M
\end{aligned}
$$



Moment of Inertia
(or second moment of an area)

Moment of inertia is the property of a deformable body that determines the moment needed to obtain a desired curvature about an axis. Moment of inertia depends on the shape of the body and may be different around different axes of rotation.

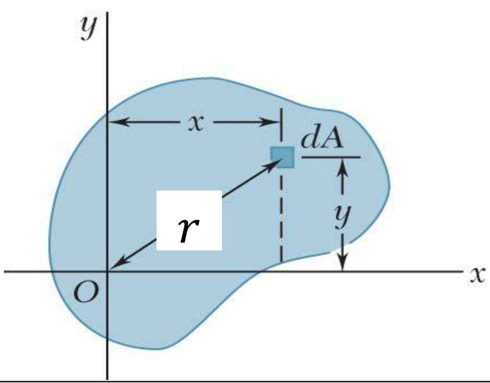


Moment-curvature relation: $|M_x| = \frac{E\,I_x}{\rho}$

E: Elasticity modulus (characterizes stiffness of the deformable body)
ρ: curvature

- The moment of inertia of the area A with respect to the x-axis is given by $I_x = \int_A y^2\,dA$
- The moment of inertia of the area A with respect to the y-axis is given by $I_y = \int_A x^2\,dA$
- Polar moment of inertia

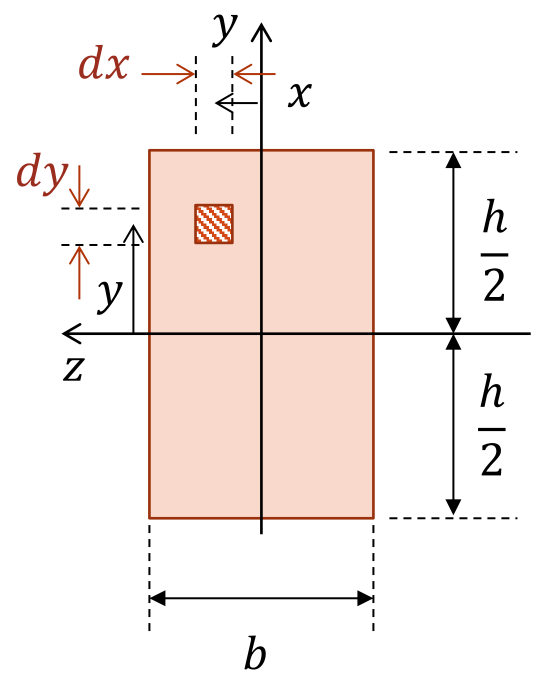
$$J = \int_A r^2\,dA = \int_A (x^2 + y^2)\,dA = I_y + I_x$$



Moment of inertia of a rectangular area

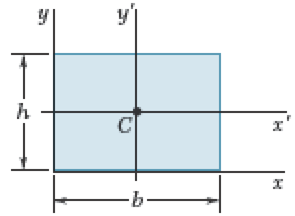
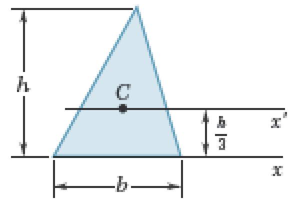
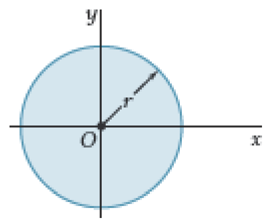
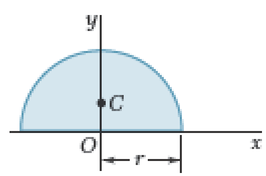
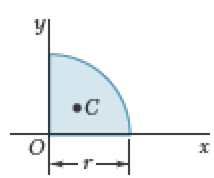
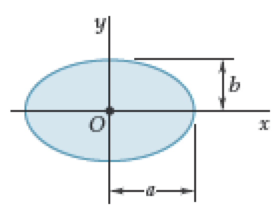
$$
\begin{aligned}
I_x &= \int_A y^2\, dA \\
&= \int_{-h/2}^{h/2} \int_{-b/2}^{b/2} y^2\, dx\, dy \\
&= \int_{-h/2}^{h/2} b\, y^2\, dy = \frac{b\, y^3}{3} \Big|_{-h/2}^{h/2} \\
&= \frac{b}{3} \left((h/2)^3 - (-h/2)^3 \right) \\
&= \frac{b}{3} \left(\frac{2h^3}{8} \right) \\
&= \frac{bh^3}{12}
\end{aligned}
$$

$$
\begin{aligned}
I_y &= \int_A x^2\, dA \\
&= \int_{-b/2}^{b/2} \int_{-h/2}^{h/2} x^2\, dy\, dx \\
&= \int_{-b/2}^{b/2} h\, x^2\, dx = \frac{h\, x^3}{3} \Big|_{-b/2}^{b/2} \\
&= \frac{h}{3} \left((b/2)^3 - (-b/2)^3 \right) \\
&= \frac{h}{3} \left(\frac{2b^3}{8} \right) \\
&= \frac{hb^3}{12}
\end{aligned}
$$



Polar moment of inertia of a circle

$$
\begin{aligned}
J_o &= \int r^2\, dA = \int_0^{2\pi} \int_0^R r^2\, (r\, dr\, d\theta) \\
&= \int_0^{2\pi} \frac{R^4}{4} d\theta = \frac{\pi R^4}{2}
\end{aligned}
$$

Rectangle		$\bar{I}_{x'} = \frac{1}{12}bh^3$ $\bar{I}_{y'} = \frac{1}{12}b^3h$ $I_x = \frac{1}{3}bh^3$ $I_y = \frac{1}{3}b^3h$ $J_C = \frac{1}{12}bh(b^2 + h^2)$
Triangle		$\bar{I}_{x'} = \frac{1}{36}bh^3$ $I_x = \frac{1}{12}bh^3$
Circle		$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$ $J_O = \frac{1}{2}\pi r^4$
Semicircle		$I_x = I_y = \frac{1}{8}\pi r^4$ $J_O = \frac{1}{4}\pi r^4$
Quarter circle		$I_x = I_y = \frac{1}{16}\pi r^4$ $J_O = \frac{1}{8}\pi r^4$
Ellipse		$\bar{I}_x = \frac{1}{4}\pi ab^3$ $\bar{I}_y = \frac{1}{4}\pi a^3b$ $J_O = \frac{1}{4}\pi ab(a^2 + b^2)$

Parallel axis theorem

- Often, the **moment of inertia** of an area is known for an axis passing through the **centroid**; e.g., x' and y':
- The moments around other axes can be computed from the known $I_{x'}$ and $I_{y'}$:

$$I_x = \int_{\text{area}} (y' + d_y)^2 \, dA$$

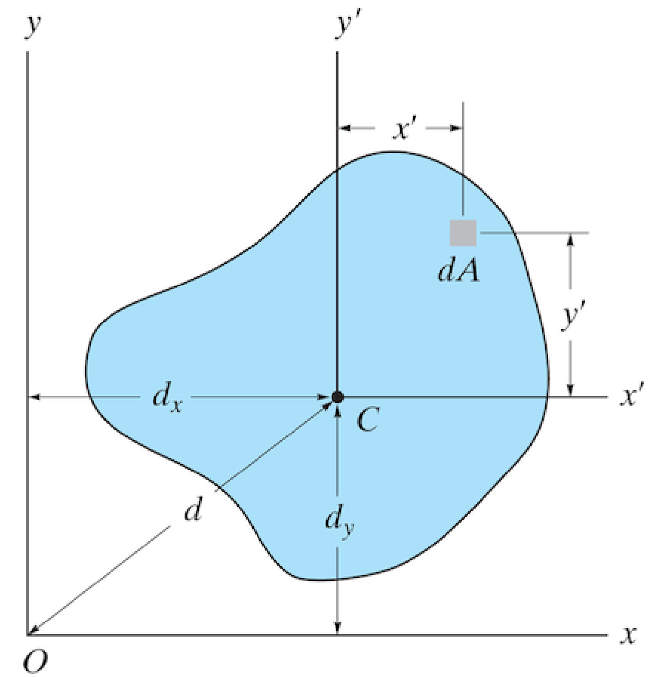
$$= \int_{\text{area}} (y')^2 \, dA + 2d_y \int_{\text{area}} y' \, dA$$

$$+ d_y^2 \int_{\text{area}} dA$$

$$= I_{x'} + Ad_y^2$$

$$I_y = I_{y'} + Ad_x^2$$

$$J_O = J_C + A(d_x^2 + d_y^2) = J_C + Ad^2$$

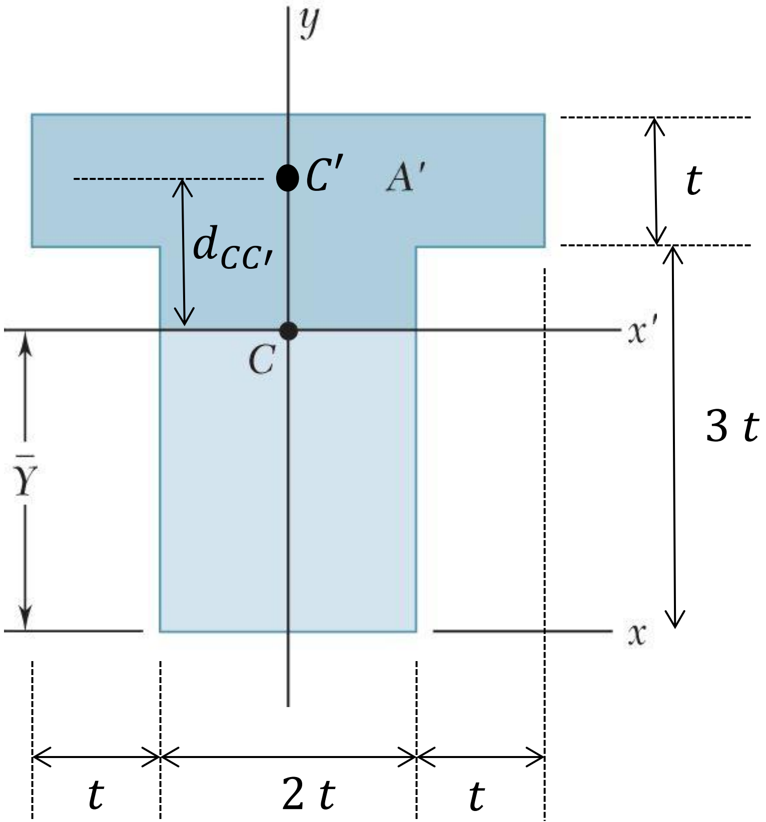


Note: the integral over y' gives zero *when done through the centroid axis*.

Moment of inertia of composite

- If individual bodies making up a **composite** body have individual areas *A* and moments of inertia *I* computed through their centroids, then the **composite area** and **moment of inertia** is a sum of the individual component contributions.
- This requires the **parallel axis theorem**
- Remember:
  - The position of the centroid of each component **must** be defined with respect to the **same origin**.
  - It is allowed to consider **negative areas** in these expressions. Negative areas correspond to holes/missing area. **This is the one occasion to have negative moment of inertia**.

From last chapter: Centroid position of the area below is given by

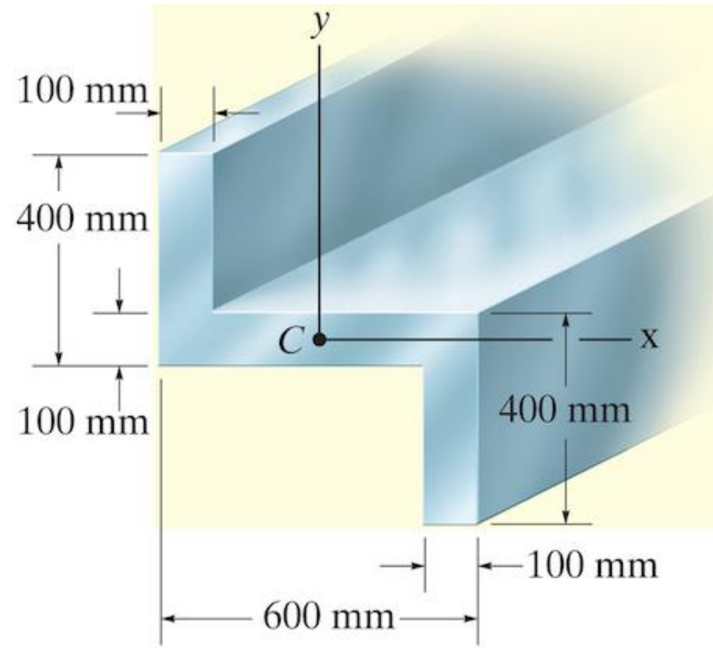


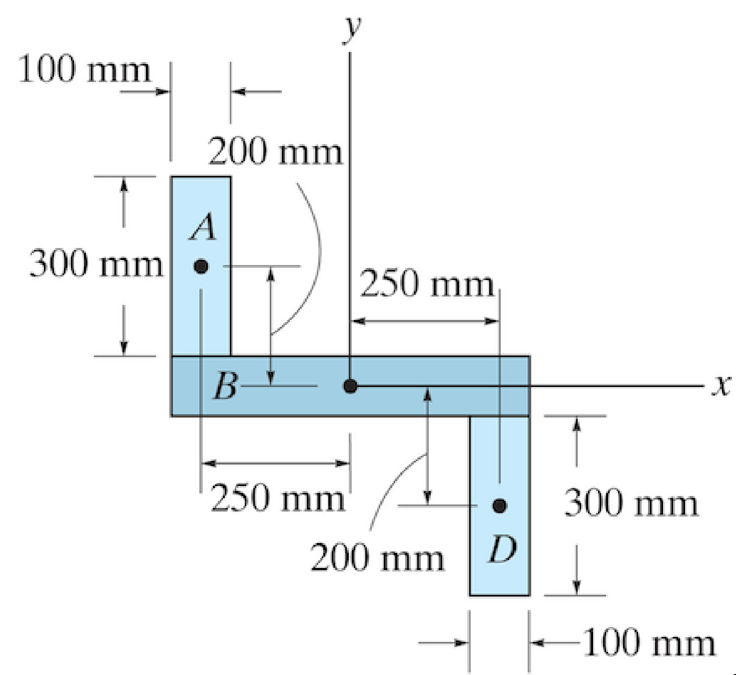
$$A_{total}\,\bar{Y} = \sum_i A_i\,\bar{y}_i$$

$$\bar{Y} = \frac{4t^2\,(3.5t) + 6t^2\,(1.5t)}{4t^2 + 6t^2} = \frac{23t}{10}$$

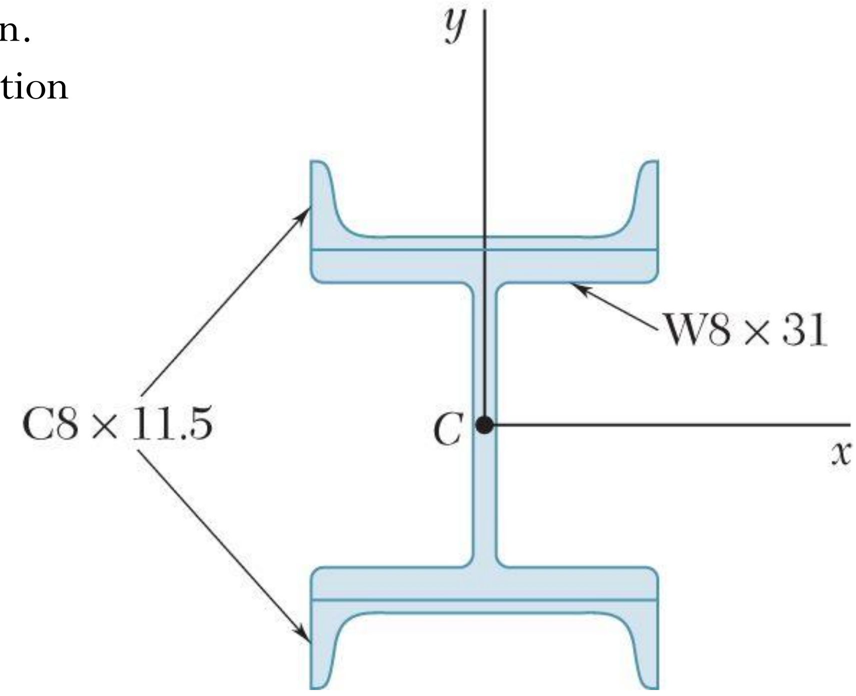
Find the moment of inertia:

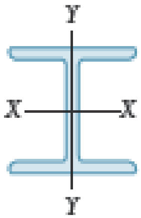
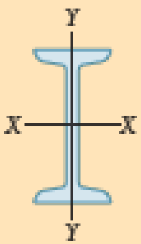
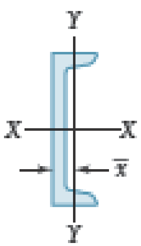
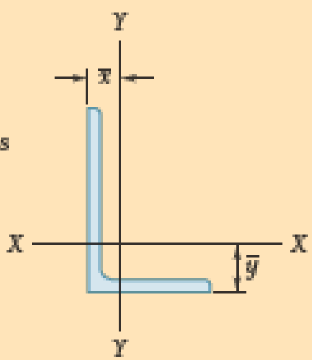
Determine the moment of inertia for the cross-sectional area about the x and y centroidal axes.

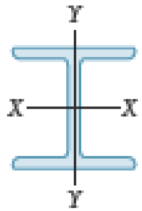
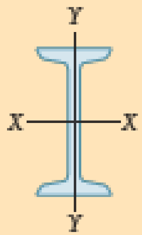
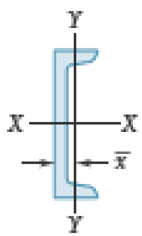
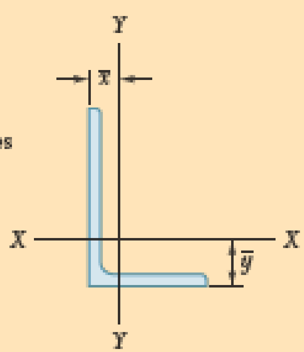




Two channels are welded to a rolled W section as shown. Determine the moments of inertia of the combined section with respect to the centroidal x and y axes.



	Designation	Area, in²	Depth, in.	Width, in.	Axis X-X			Axis Y-Y		
					$\bar{I}_x$, in⁴	$\bar{k}_x$, in.	$\bar{y}$, in.	$\bar{I}_y$, in⁴	$\bar{k}_y$, in.	$\bar{x}$, in.
W Shapes (Wide-Flange Shapes) 	W18 × 76†	22.3	18.2	11.0	1330	7.73		152	2.61	
	W16 × 57	16.8	16.4	7.12	758	6.72		43.1	1.60	
	W14 × 38	11.2	14.1	6.77	385	5.87		26.7	1.55	
	W8 × 31	9.12	8.00	8.00	110	3.47		37.1	2.02	
S Shapes (American Standard Shapes) 	S18 × 54.7†	16.0	18.0	6.00	801	7.07		20.7	1.14	
	S12 × 31.8	9.31	12.0	5.00	217	4.83		9.33	1.00	
	S10 × 25.4	7.45	10.0	4.66	123	4.07		6.73	0.980	
	S6 × 12.5	3.66	6.00	3.33	22.0	2.45		1.80	0.702	
C Shapes (American Standard Channels) 	C12 × 20.7†	6.08	12.0	2.94	129	4.61		3.86	0.797	0.698
	C10 × 15.3	4.48	10.0	2.60	67.3	3.87		2.27	0.711	0.634
	C8 × 11.5	3.37	8.00	2.26	32.5	3.11		1.31	0.623	0.572
	C6 × 8.2	2.39	6.00	1.92	13.1	2.34		0.687	0.536	0.512
Angles 	L6 × 6 × 1‡	11.0			35.4	1.79	1.86	35.4	1.79	1.86
	L4 × 4 × $\frac{1}{2}$	3.75			5.52	1.21	1.18	5.52	1.21	1.18
	L3 × 3 × $\frac{1}{4}$	1.44			1.23	0.926	0.836	1.23	0.926	0.836
	L6 × 4 × $\frac{1}{2}$	4.75			17.3	1.91	1.98	6.22	1.14	0.981
	L5 × 3 × $\frac{1}{2}$	3.75			9.43	1.58	1.74	2.55	0.824	0.746
	L3 × 2 × $\frac{1}{4}$	1.19			1.09	0.953	0.980	0.390	0.569	0.487

	Designation	Area mm^2	Depth mm	Width mm	Axis X-X			Axis Y-Y		
					$\bar{I}_x$ 10^6 mm^4	$\bar{k}_x$ mm	$\bar{y}$ mm	$\bar{I}_y$ 10^6 mm^4	$\bar{k}_y$ mm	$\bar{x}$ mm
W Shapes (Wide-Flange Shapes) 	W460 × 113†	14400	462	279	554	196		63.3	66.3	
	W410 × 85	10800	417	181	316	171		17.9	40.6	
	W360 × 57.8	7230	358	172	160	149		11.1	39.4	
	W200 × 46.1	5880	203	203	45.8	88.1		15.4	51.3	
S Shapes (American Standard Shapes) 	S460 × 81.4†	10300	457	152	333	180		8.62	29.0	
	S310 × 47.3	6010	305	127	90.3	123		3.88	25.4	
	S250 × 37.8	4810	254	118	51.2	103		2.80	24.1	
	S150 × 18.6	2360	152	84.6	9.16	62.2		0.749	17.8	
C Shapes (American Standard Channels) 	C310 × 30.8†	3920	305	74.7	53.7	117		1.61	20.2	17.7
	C250 × 22.8	2890	254	66.0	28.0	98.3		0.945	18.1	16.1
	C200 × 17.1	2170	203	57.4	13.5	79.0		0.545	15.8	14.5
	C150 × 12.2	1540	152	48.8	5.45	59.4		0.286	13.6	13.0
Angles 	L152 × 152 × 25.4‡	7100			14.7	45.5	47.2	14.7	45.5	47.2
	L102 × 102 × 12.7	2420			2.30	30.7	30.0	2.30	30.7	30.0
	L76 × 76 × 6.4	929			0.512	23.5	21.2	0.512	23.5	21.2
	L152 × 102 × 12.7	3060			7.20	48.5	50.3	2.59	29.0	24.9
	L127 × 76 × 12.7	2420			3.93	40.1	44.2	1.06	20.9	18.9
	L76 × 51 × 6.4	768			0.454	24.2	24.9	0.162	14.5	12.4